

A NOTE ON THE DERIVATIONS OF TWO KNOWN COMBINATORIAL IDENTITIES VIA A HYPERGEOMETRIC SERIES APPROACH

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ABSTRACT. The aim of this note is to derive two known combinatorial identities via a hypergeometric series approach using Saalschitz's classical summation theorem.

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by [8]:

$$(1.1) \quad {}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

where $(a)_n$ denotes the well-known pochhammer symbols (or the raised or the shifted factorial, since $(1)_n = n!$) defined for any complex number $a (\neq 0)$ by

$$(a)_n = \begin{cases} a(a+1) \cdots (a+n-1), & n \in \mathbb{N} \\ 1, & n = 0. \end{cases}$$

Also, in terms of well-known Gamma functions, Pochhammer symbol $(a)_n$ is given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Here, as usual, p and q are non-negative integers and the parameters a_j ($1 \leq j \leq p$) and b_j ($1 \leq j \leq q$) can take arbitrary values (real of

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complex) with one exception that b_j ($1 \leq j \leq q$) should not be zero or a negative integer.

By well-known ratio tests, it can be easily verified that the generalized hypergeometric function ${}_pF_q(z)$ converges for all $|z| < \infty$ ($p \leq q$), $|z| < 1$ ($p = q + 1$) and $|z| = 1$ ($p = q + 1$) with $\Re(s) > 0$, where s is the parametric excess defined by

$$s = \sum_{j=1}^q b_j - \sum_{j=1}^p a_j.$$

For $p = 2$ and $q = 1$, we have the well-known hypergeometric function ${}_2F_1(z)$ popularly known in the literature as the Gauss's hypergeometric function.

It is not out of place to mention here that the generalized hypergeometric function occurs in many theoretical and practical applications in the areas of mathematics, theoretical physics, mathematical physics, statistics, and engineering mathematics.

For more details about the generalized hypergeometric functions ${}_pF_q(z)$ about its convergence conditions, properties and connection with several elementary functions, we refer the standard texts [1, 2, 6].

It is interesting to mention here that whenever a generalized hypergeometric function reduces to the gamma function, the results are very important from the applications point of view. Thus the classical summation theorems such as those of Gauss, Gauss-second, Bailey and Kummer for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschitz for the series ${}_3F_2$ and others play an important role. However, in our present investigation, we would like to mention here the following classical Saalschitz [8] summation theorem for the series ${}_3F_2$ viz.

$$(1.2) \quad {}_3F_2 \left[\begin{matrix} -n, a, b \\ c, 1 + a + b - c - n \end{matrix} ; 1 \right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n},$$

REMARK 1.1. For generalizations and extensions of the above-mentioned classical summation theorems, we refer research papers by Rakha and Rathie [7] and Kim et al. [4].

On the other hand, we mention here the following two well-known combinatorial identities recorded in Gould [3, Equ. (3.96) and (3.95), p.33] and Sprugnoli [9, Equ. (3.95) and (3.96), p.77] viz.

$$(1.3) \quad \sum_{k=0}^n \frac{1}{2k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{2^{4n}}{(2n+1)} \binom{2n}{n} = \frac{2^{2n}(1)_n}{\left(\frac{3}{2}\right)_n}$$

and

$$(1.4) \quad \sum_{k=0}^n \frac{1}{k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = \binom{2n+1}{n} = \frac{2^{2n} \left(\frac{3}{2}\right)_n}{(2)_n}.$$

Very recently Qi et al. [5] established the identity (1.3) by two different methods using series expansion of powers of arcsine and also discussed the identity (1.4).

The aim of this short note is to establish the combinatorial identities (1.3) and (1.4) by using classical Saalschütz summation theorem (1.2).

2. Derivations of two combinatorial identities (1.3) and (1.4) using Saalschütz summations Theorem (1.2)

2.1. Derivation of the identity (1.3)

In order to derive the identity (1.3), we proceed as follows. Denoting the left-hand side of the identity (1.3) by S_1 , we have

$$(2.1) \quad S_1 = \sum_{k=0}^n \frac{1}{2k+1} \binom{2k}{k} \binom{2n-2k}{n-k}.$$

Now making use of the following elementary results and identities:

$$(i) \quad a = \frac{\Gamma(a+1)}{\Gamma(a)}$$

$$(ii) \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$(iii) \quad \Gamma(a-n) = \frac{(-1)^n \Gamma(a)}{(1-a)_n}$$

$$(iv) \quad \Gamma(2z) = \frac{2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2})}{\sqrt{\pi}}$$

$$\text{and } (v) \quad \binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1) \Gamma(n-k+1)}.$$

It is not difficult to see that

$$\begin{aligned} (a) \quad 2k + 1 &= \frac{\left(\frac{3}{2}\right)_k}{\left(\frac{1}{2}\right)_k} \\ (b) \quad \binom{2k}{k} &= \frac{2^{2k} \left(\frac{1}{2}\right)_k}{k!} \\ \text{and } (c) \quad \binom{2n-2k}{n-k} &= \frac{2^{2n-2k} (-n)_k \left(\frac{1}{2}\right)_k}{(1)_n \left(\frac{1}{2} - n\right)_k}. \end{aligned}$$

Substituting all the expressions in (2.1), we have after some simplification

$$S_1 = \frac{2^{2n} \left(\frac{1}{2}\right)_n}{(1)_n} \sum_{k=0}^n \frac{(-n)_k \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k \left(\frac{1}{2} - n\right)_k k!}.$$

Finally, summing up the series, we have

$$S_1 = \frac{2^{2n} \left(\frac{1}{2}\right)_n}{(1)_n} {}_3F_2 \left[\begin{matrix} -n, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{1}{2} - n \end{matrix} ; 1 \right].$$

We now observe that the ${}_3F_2$ can be evaluated with the help of the classical Saalschütz summation theorem (1.2) by letting $a = b = \frac{1}{2}$ and $c = \frac{3}{2}$. We thus at once arrive at the right-hand side of (1.3). This completes the proof of combinatorial identity.

2.2. Derivation of the identity (1.4)

Denoting the left-hand side of (1.4) by S_2 and proceeding exactly on the similar lines as given in (2.1), we, after some simplification arrive at the following expression

$$S_2 = \frac{2^{2n} \left(\frac{1}{2}\right)_n}{(1)_n} {}_3F_2 \left[\begin{matrix} -n, \frac{1}{2}, 1 \\ 2, \frac{1}{2} - n \end{matrix} ; 1 \right].$$

The ${}_3F_2$ can be evaluated by Saalschütz summation theorem (1.2) by letting $a = \frac{1}{2}$, $b = 1$ and $c = 2$. So we left this as an exercise to the interested reader.

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